Onset of congestion due to low speed merging maneuvers within a free-flow traffic stream: Analytical solution

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Abstract

Low speed merging maneuvers performed within a free-flow stream are believed to trigger congestion. These accelerating moving bottlenecks introduce local constraints that can disturb the flow at a local or at a global scale. Low speed merging maneuvers are also suspected to cause capacity drop.

Based on the kinematic wave theory, this paper explores the analytical solution of a simple first order model when moving boundary conditions are introduced.

The paper shows that shockwaves initiated by low speed merging maneuvers are a linear transformation of the moving boundary conditions, no matter the shape of the moving boundary condition. These results are then applied on typical situations to show that the interaction of two moving boundaries can modify the analytical solution of the problem.

The results are then extended to multiple merging maneuvers, showing that they can interact. Every possible interaction between two identical merging maneuvers is explored to identify the conditions that lead to global congestion. Finally these results are used to propose an analytical formulation of the capacity drop for multiple merging maneuvers at a single location. We show that it is related to the demands on the minor and the major stream and to speed of the merging vehicle.

Table of Symbols

v_f	Free-flow speed	m.s^{-1}
s_x	Minimum distance at zero speed	m
k_x	Maximum density at zero speed	m^{-1}
w	Maximum shockwave's speed	m.s^{-1}
k_c	Critical density	m^{-1}
q_c	Capacity on a single lane	veh.s^{-1}
v_0	Merging speed	m.s^{-1}
T^f	Free-flow trajectory	-
Γ	Moving Boundary Condition (MBC) that represents a bounded acceleration trajectory	-
$\Theta(v_0, v_f)$	Time required for a vehicle merging at the speed v_0 to reach the speed v_f	s
$\mathrm{X}(v_0, v_f)$	Distance required for a vehicle merging at the speed v_0 to reach the speed v_f	m
W	Shockwave resulting from a MBC Γ	-
$\Delta T_0^N(X)$	Time interval between the vehicle *0* and the vehicle *N* at the location *X*	s
$Q_{0\rightarrow N}(X)$	Mean flow between the vehicle *0* and the vehicle *N* at the location *X*	veh.s^{-1}
γ	Ratio representing the amplitude of a capacity-drop	-

1. Introduction

Traffic congestion is a major problem in our modern society. Besides increasing travel time, it leads to serious externalities such as increasing fuel consumption, gas emission and noise pollution. If its propagation is now well-understood and integrated in the current theory, the onset of congestion is still a challenging question for the traffic flow theory community.

1.a Onset of the congestion on motorways : state of the art

Observation shows that the onset of congestion takes place around discontinuities (within weaving sections, near ramps, etc.). An example of such a discontinuity is a freeway where vehicles from the minor stream merge into the major stream thereby modifying the road's capacity (Elefteriadou et al, 1995). The capacity is defined as the maximum possible flow passing through the downstream point of the merge. Elefteriadou (Elefteriadou et al, 2006) measured three different capacities (in free-flow regime; at the time of breakdown; during the queue discharge), underlying the concept of dynamic capacity. In the last two decades empirical observations have also shed light on the capacity drop phenomenon (Hall et al, 1991, Persaud et Al, 1998, Laval, et al., 2005, Chung et al. 2007). Magnitude level of the capacity drop is around 15 percents according to some bottleneck studies (Elefteriadou et al., 1995; Cassidy and Bertini, 1999; Bertini and Leal, 2005).

Recent studies have shown that discretional and mandatory lane-changing (LC) maneuvers are a primary factor for the onset of congestion and that the amplitude of oscillation is related to the lane-changing activity (Cassidy and Rudjanakanoknad, 2005; Ahn and Cassidy, 2007). In this context, control strategies have been developed to avoid or prevent congestion. For example, ramp metering strategies smooth the demand from the minor stream and increase the speed of the vehicle that merge into the main stream. It leads to a significant positive effect by decreasing up to 20% the total time spent in the merging area (Haj-Salem et al, 1995, Papageorgiou et al, 2007). This sheds light on the link between the speed at the merging maneuver and the onset of congestion. A first explanation of this phenomenon was proposed in (Coifman, 2006): LC maneuvers can create voids within the stream or lead to a non-equilibrium situation within the traffic stream.

Some tools have been developed to measure the effect of LC maneuvers in congested regime. Coifman et al (2006) proposed an approach based on individual delays' measurements from trajectory data. But the methodology does not consider the relaxation phenomenon and therefore cannot be used for a microscopic approach. Furthermore Wang and Coifman (2008) studied the effect of LC maneuvers on speed spacing. Duret et al. (2009) proposed a methodology for measuring the effect of LC maneuvers on a traffic stream. All the methodologies proposed in these papers cannot explain the dual capacity phenomenon (arising during the onset of congestion) from a first order kinematic wave theoretical point of view.

Even so the capacity drop is a major issue that must be reproduced by our traffic models. First order models (Lighthill and Whitham, 1955, Richards, 1956) have been initially developed for uninterrupted flow. The kinematic wave theory is efficient in order to model the uninterrupted flow since boundary conditions are well defined. Its analytical solutions are fully consistent with observation (like shockwave propagation) and are easy to compute for a wide variety of simple but nontrivial cases. However, Mauch and Cassidy (2002) have shown that Newell's model (Newell, 1993) fails for reproducing the effect of heavy lane-changing activity on queued freeway. Indeed, some assumptions of this model are inconsistent with observation: the speed adjusts instantaneously which results in unrealistic infinite acceleration or deceleration. Therefore this model is not appropriate for modeling traffic around discontinuities and does not reproduce some phenomenon such as capacity drop.

Consequently, the second order models have been developed in an attempt to fill this gap. In the second order model proposed by Payne (1971), the conservation law is completed with an equation that reflects the evolution of the flow speed. Analytical solutions of these models are complicated and present some inconstancies. Subsequently, they have faced a number of comparisons with first order models (Lebacque and Lesort, 1999) and criticisms (Del Castillo et al 1994, Daganzo, 1995). The main critic lies on the fact that under some initial condition, second order models can lead to negative flows, and can also lead to wave speeds that can be higher than the mean speed of simulated traffic. Furthermore, a new version of these models, that avoids negative flows, was proposed in (Zhang, 2002). Commonly, second order models are able to reproduce macroscopic phenomenon like relaxation and capacity drop.

Recently, some complementary models have been developed based on the first order model, like the bounded acceleration model (Leclercq, 2007a) and the lane-changing model (Laval and Daganzo, 2006). The mechanism of LC maneuvers under congested regime is now taken into account within first order model (Laval and Leclercq, 2008, Wang and Coifman, 2008). On this basis, (Duret et al., 2009, 2010) has developed a methodology and show that the impact of LC maneuvers is null under congested regime. The relaxation plays an important part within a congested stream and the literature provides a comprehensive understanding of its mechanism and impact. But that is not the case for merging maneuvers with low speeds within a free-flow stream close to the capacity. Recently, (Laval, 2005) explained that the capacity drop can be due to vehicles from a minor stream that insert within the major stream at low speed. The author combines a simple first order model combined with a LC model and shows that the model is able to reproduce the capacity drop. This result is not surprising since the LC model introduces a Moving Boundary Condition (MBC) that modifies the flow properties.

(Daganzo and Laval, 2005) have presented a numerical method to model kinematic wave traffic stream containing slow vehicles. Here we study a numerical method to model similar stream containing vehicles that insert at low speed and then accelerate to reach the free-flow speed. These vehicles represent MBCs. Note that the results of this paper can be extended to any MBCs.

The aim of this article is twofold.

- Express the analytical solution of low speed merging maneuvers within a free-flow stream
- Propose an explicit formulation of the capacity-drop

The literature provides a lot of car-following models, and Brackstone et al. (1999) is recommended for an overview. The car-following model proposed by Newell (1993, 2002) is used for its simplicity and its ability to reproduce basic traffic features (Ahn et al. 2004, Duret et al., 2008).

1.b Modelling framework: the Newell model

The Newell model is based on the conservation law that, in Eulerian coordinates fits

$$\frac{\partial k}{\partial t}+\frac{\partial Q(k)}{\partial x}=0 \tag{1}$$

where k is the density and $Q(k)$ a flux function that describes the flow as a function of the density. It is also called fundamental diagram (FD). The solution of this hyperbolic equation is known when assuming the concavity of the FD (Daganzo, 2005a, 2005b).
The Newell CF model corresponds to the LWR model (Lighthill and Whitham, 1955, Richards, 1956) when applied with a triangular FD (Newell, 1993, Newell, 2002). It only holds three parameters with a physical meaning: the free-flow speed v_f, the maximum shockwave speed w and the jam density k_x (Figure 1(a)). This triangular shape is supported by empirical observations and has been verified on microscopic data (Ahn et al. 2004, Duret et al., 2008, Chiabaut et al., 2009)

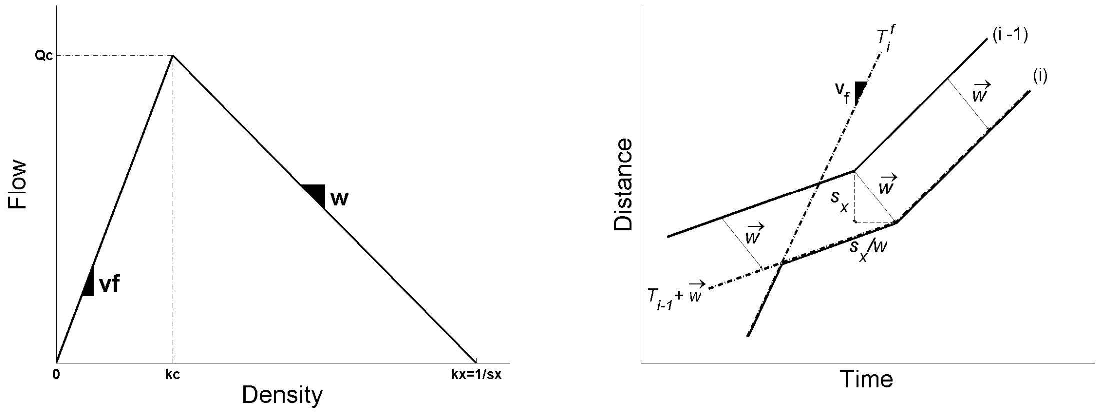
The solution of this model is straightforward and exact when applied on a relevant Lagrangian numerical scheme (Leclercq, 2007b). The trajectory of a vehicle i is given as the compromise for travelling at its free-flow speed (*demand)* without passing over a constraint from the vehicle i-1 (*supply*). Here the constraint corresponds to the leader's trajectory but shifted in time and space with a vector $\vec{w}=[\frac{s_x}{w};s_x]$ (see figure 1(b)). As a consequence, the position $x_i^{t+\Delta T}$ of vehicle i at time $t+\Delta T$ is

$$x_i^{t+\Delta T}=\min\left(\underbrace{x_i^t+v_f.\Delta T}_{demand}\quad,\quad\underbrace{x_{i-1}^{t+\Delta T-\frac{sx}{w}}-s_x}_{supply}\right)$$

and the trajectory T_i of the vehicle i is

$$T_i=\min\left(\underbrace{T_i^{\,f}}_{demand}\quad,\quad\underbrace{T_{i-1}+\vec{w}}_{supply}\right) \tag{2}$$

where $T_i^{\,f}$ is the upstream demand trajectory and $T_{i-1}+\vec{w}$ corresponds to the downstream supply trajectory.



(a) **(b)**

Figure 1 Newell CF model (a) triangular FD (b) piecewise linear vehicle trajectories

This theory, also called kinematic wave theory, is built on assumptions inconsistent with observations such as finite acceleration. To fill this gap, some bounded acceleration models have been proposed (Lebacque (1997), Giorgi et al. (2002)) to consider a finite acceleration that can be fixed or variable. These bounded accelerations are integrated in the Newell model through the introduction of MBCs. Note that we disregard the relaxation effect that can occur in specific free-flow situations near capacity.

1.c Objectives and content of the paper

The aim of this paper is to present the analytical solutions of the kinematic wave model when low speed vehicles merge within a free-flow stream. These vehicles accelerate to reach the free-flow speed and their trajectories represent some MBCs, which is denoted by Γ_0.

In the remainder of the paper, the MBCs will not be related to a specific bounded acceleration model and the results can be extended to any MBCs. $\Theta(v_0, v_f)$ and $X(v_0, v_f)$ will refer to the time and the distance to go from the merging maneuver speed v_0 to the free-flow speed v_f. They will be respectively denoted by Θ and X. Moreover, the following assumptions will always be satisfied:

- a constant upstream demand Q_A and $t_A = 1/Q_A$ a constant time interval between two vehicle arrivals on the targeted lane;
- a triangular FD;
- a known MBC Γ_0 (i.e the acceleration model);
- identical FD and acceleration model for every vehicle.

2. Analytical solution

This section presents the analytical solution of the model. The first part considers a single merging maneuver and presents the condition for initiating a shockwave that propagates in space and time. Then this condition is considered as satisfied and we demonstrate that the shockwave is a linear transformation of the MBCs. Finally, the analytical solution of the problem is presented considering different possible interactions between two merging maneuvers.

2.1 Single merging maneuver

2.1.a Condition for initiating a shockwave

Consider here a single merging maneuver in a single lane. Vehicles are consecutively numbered from the reference vehicle that inserts *(0)* to its n[th] follower *(n)*. Vehicle *0* inserts at the speed $v_0 < v_f$ at the reference time $t = 0$ and space $x = 0$. Vehicle *0* accelerates and reaches the speed v_f at the time $t = \Theta$ and the location $x = \mathrm{X}$; its trajectory causes a MBC, denoted by Γ_0 (figure 2).
This merging maneuver may have an impact on the upstream traffic depending on whether the vehicle *1* switches to a congested regime; i.e. the upstream demand is not constrained by the downstream supply.
Here the demand for vehicle *1* corresponds to its free-flow trajectory T_1^f and the formula for the supply is $\Gamma_1 = \Gamma_0 + \vec{w}$. The kinematic wave theory (equation (2)) gives that the final trajectory T_1 of the vehicle *1* is $\mathrm{T}_1 = \min(\underbrace{\mathrm{T}_1^f}_{demand}, \underbrace{\Gamma_1}_{supply})$, as illustrated in figure 2.

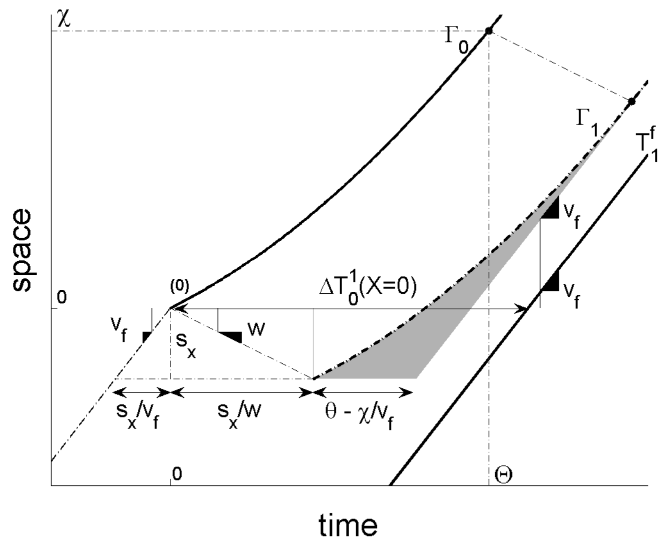


Figure 2 Trajectories of the vehicle *0* (Γ_0) and the vehicle *1*(T_i^f)

In this figure, $\Delta T_0^1\left(X=0\right)$ represents the initial time interval between the vehicle *0* and the vehicle *1* at the point $X=0$ (note that $\Delta T_0^1\left(X=0\right)<1/Q_A$). It demonstrates that the merging maneuver has no impact if $\Delta T_0^1\left(X=0\right)$ fits:

$$\Delta T_0^1\left(X=0\right)>\left(\frac{s_x}{w}+\frac{s_x}{v_f}\right)+\left(\Theta-\frac{\mathrm{X}}{v_f}\right)=\frac{1}{Q_c}+\left(\Theta-\frac{\mathrm{X}}{v_f}\right) \quad (3)$$

This condition ensures that the merging maneuver of the vehicle *0* does not disturb vehicle *1* (and consequently any of its followers). Note that this condition is respected for some combination of the merging maneuver parameter v_0 (that determines Θ and X), the FD parameters (v_f and Q_C) and Q_A.

The remainder of this section describes the analytical solution of this problem when this condition is not satisfied.

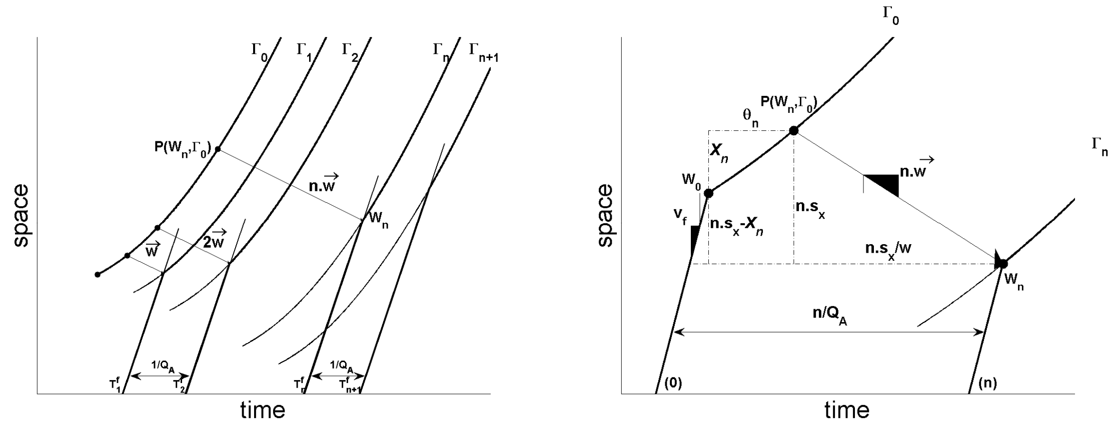
2.1.b Notation

In the remainder of the paper the condition (equation (3)) is not satisfied. As a consequence, a single merging maneuver initiates a shockwave that propagates in time and space. This section shows that the shockwave's trajectory, denoted W, is a linear transformation of the boundary condition Γ_0; the function that represents this transformation combines the FD parameters (v_f, Q_c, K_x) and the demand (Q_A).

Consider that the condition from equation (3) is not satisfied. One or more vehicles following the merging maneuver are impacted. Vehicle *n* (impacted) has the upstream demand T_n^f and the downstream supply $\Gamma_n=\Gamma_0+n.\vec{w}$. According to the equation (2), this vehicle's final trajectory T_n is determined by the upstream demand and the downstream supply $\mathrm{T}_n=\min(\underbrace{\mathrm{T}_n^f}_{demand},\underbrace{\Gamma_n}_{supply})$, as illustrated in figure 3(a).

Let $W_n=\begin{bmatrix}\theta_n^{'}\\ \chi_n^{'}\end{bmatrix}$ be the intersection between T_n^f and Γ_n. It is convenient to introduce the function P that projects the point $\begin{bmatrix}\theta_n\\ \chi_n\end{bmatrix}$ along the MBC Γ_0 with a constant slope *w*.

$P(W_n,\Gamma_0)=\begin{bmatrix}\theta_n\\ \chi_n\end{bmatrix}$, as illustrated in figure 3(a).



(a) **(b)**

Figure 3 Vehicles impacted by a single merging maneuver: (a) Trajectories of the vehicle *0* to *n* (b) Detailed scheme of the vehicles *0* and *n*

Note that, by definition, W_n belongs to Γ_n and T_n^f. As a consequence, W_n satisfies some conditions.

As, W_n belongs to Γ_n, according to the kinematic wave theory, W_n is a simple shift of $P(W_n, \Gamma_0)$ with a vector $n.\vec{w}$. As illustrated in figure 3(a), it follows

$$\begin{bmatrix} \theta_n' \\ \chi_n' \end{bmatrix} = \begin{bmatrix} \theta_n \\ \chi_n \end{bmatrix} + n.\vec{w} \qquad (4.a)$$

W_n also belongs to T_n^f . The time headway between the vehicle *0* and the vehicle *n* is $\Delta T_0^n(X = \chi_n) = n / Q_A$. Graphically, we can see that the time headway also satisfies $\Delta T_0^n(X = \chi_n) = \theta_n + \frac{n.s_x - \chi_n}{v_f} + \frac{n.s_x}{w}$ (see figure 3(b)). It follows

$$\frac{n}{Q_A} = \theta_n + \frac{n.s_x - \chi_n}{v_f} + \frac{n.s_x}{w} \qquad (4.b)$$

This formula gives a general expression of *n*:

$$n = \left(\frac{1}{Q_A} - \frac{1}{Q_C} \right)^{-1} \left(\theta_n - \frac{\chi_n}{v_f} \right) \qquad (5)$$

Note that if N represents the total number of vehicles impacted by a single merging maneuver, then the following holds:

$$N = \left(\frac{1}{Q_A} - \frac{1}{Q_C} \right)^{-1} . \left(\Theta - \frac{\mathrm{X}}{v_f} \right) \qquad (6)$$

It is interesting to note that *N* only depends on three factors:

- the demand in traffic Q_A
- the fundamental diagram parameters v_f and Q_c
- the acceleration parameters X and Θ

N is independent from the boundary shape (i.e. the dynamic of the acceleration) but only depends on the time-distance Θ and the space-distance X. Note as well that when the demand equals the capacity, the shockwave initiated by a single merging maneuver cannot clear; it propagates upstream the section at the speed w and the number of vehicles impacted by a single merging maneuver tends to be infinite.

2.1.c Analytical expression of the shockwave W

The combination of equation (5) and (6) gives $\begin{bmatrix} \theta_n^{'} \\ \chi_n^{'} \end{bmatrix} = A.\begin{bmatrix} \theta_n \\ \chi_n \end{bmatrix}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left(\frac{1}{Q_A} - \frac{1}{Q_C} \right)^{-1} . \frac{1}{k_x} \begin{bmatrix} \frac{1}{v_f} & -\frac{1}{w.v_f} \\ -1 & \frac{1}{w} \end{bmatrix} \quad (7)$$

This system is met for every n such that $1 < n \leq N, n \in \mathbb{N}$ and can be extended over the total number of vehicles impacted by the merging maneuver (i.e. that cross the shockwave). Now let W denote the shockwave initiated by the merging maneuver, equation (7) yields:

$$W = \Gamma_0 + \Phi(\Gamma_0), \quad (8)$$

With

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2$$

$$X \to \left(\frac{1}{Q_A} - \frac{1}{Q_C} \right)^{-1} . \frac{1}{k_x} \begin{bmatrix} \frac{1}{v_f} & \frac{1}{w.v_f} \\ -1 & \frac{1}{w} \end{bmatrix} \quad (9)$$

Note that Φ is continuous and combines the fundamental diagram parameters (v_f, Q_c, K_x) and the demand Q_A. In the remainder of the paper, W is considered as mathematically continuous even if the physical shockwave is only defined for entire *n*-values.

Note that some this demonstration considers some assumptions about homogeneity that can be relaxed: (i) The first assumption consider a constant time headway $1/Q_A$ between two successive vehicles. It can be relaxed, considering individual time headways $1/Q_i$ in equation *(4.b)*. (ii) The second assumption identical fundamental diagram. It can be relaxed as well, considering individual free-flow speed v^f_i, individual maximum shockwave speeds w_i and individual minimum headway at stop $s_{x,i}$ in equation *(4.a)*. The combination of the new formula (4.a) and (4.b) then give the analytical solution when the assumptions about homogeneity are relaxed. This demonstration is not included in interest of brevity.

This section has described the analytical solution of the first order model with a MBC Γ_0 (a vehicle that inserts a speed lower than v_f and then gradually accelerates). It shows that under specific condition, Γ_0 initiates a shockwave W that propagates in space and time. The continuous analytical formulation of W is a simple linear transformation Φ of the MBC Γ_0; Φ is a function that depends on the demand Q_A and the FD.

On this basis, the next section extends the results for multiple merging maneuvers.

2.2 Multiple merging maneuvers

2.2.a Notations

Let's consider a section with a single lane and a multiple merging maneuver yielding multiple MBCs. The scenario meets the assumptions (i-iv) and two new notations are introduced:

- Γ_i now represents the trajectory of the merging maneuver *i*; it also represents the MBC *i*
- $\begin{bmatrix} T_i \\ X_i \end{bmatrix}$ represents the time-space location of the merging maneuver *i*; it also represents the origin of the MBC Γ_i

In this section, we explore every possible interaction between two identical successive merging maneuvers, denoted *i* and *i+1*. They are supposed to be free of any impact from other merging maneuvers.

The merging maneuver *i* introduces a MBC within the stream at the location $\begin{bmatrix} T_i \\ X_i \end{bmatrix}$ and separates the $[t, x]$ diagram into four areas, denoted A-B-C-O, as illustrated in figure 4.a. Figure 4.b show the corresponding steady-states in each area on a triangular FD.

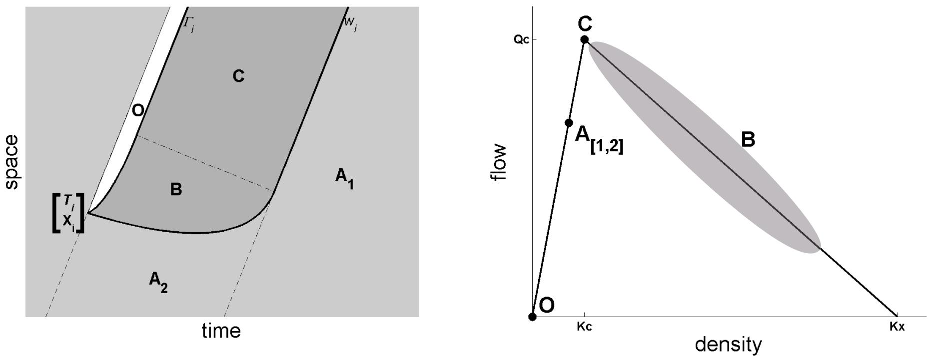


Figure 4 (a) (t,x) diagram around a single merging maneuver (b) corresponding steady-states on a triangular fundamental diagram

2.2.b Possible interactions

Consider now the merging maneuver *i+1* that occurs at the location $\begin{bmatrix} T_{i+1} \\ X_{i+1} \end{bmatrix}$. The shockwaves W_i and W_{i+1} respectively initiated by the merging maneuvers *i* and *i+1* may interact. The next section shows that the global shockwave W^* is a linear transformation of a fictitious MBC Γ^*.

It can be derived: $W^* = A(\Gamma^*)$ with $\Gamma^* = \Psi(\Gamma_i, \Gamma_{i+1})$

To illustrate this point, let's consider the four possible interactions illustrated in figure 5.

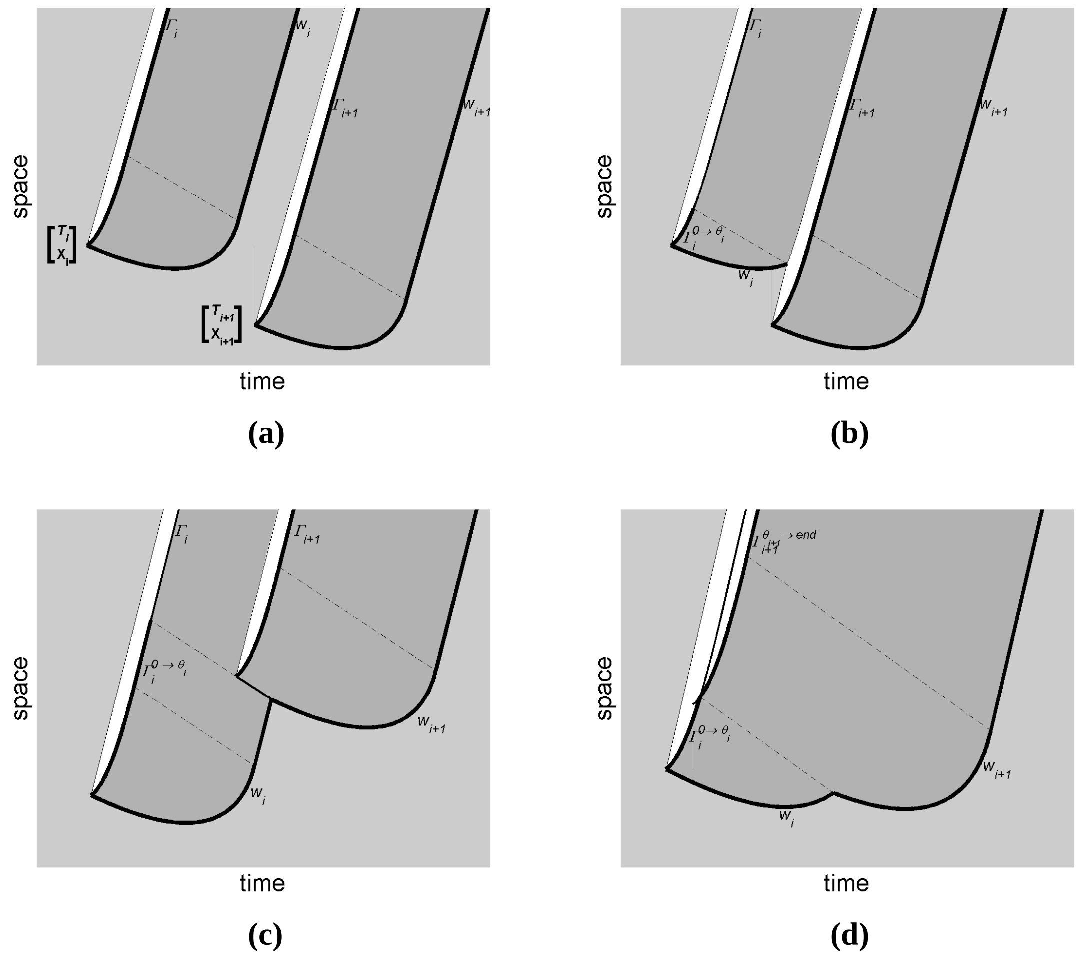


Figure 5 four possible interactions (a) case 1 (b) case 2 (c) case 3 (d) case 4

Case 1 : $\begin{bmatrix} T_{i+1} \\ X_{i+1} \end{bmatrix} \in A1$ (figure 5.a)

The merging maneuver *i+1* occurs in region $A1$. This region is not concerned with merging maneuver *i*. The two merging maneuvers do not interact and their respective impacts are independent.

$$W_i = A.(\Gamma_i) \quad \text{and} \quad W_{i+1} = A.(\Gamma_{i+1}) \qquad (10.1)$$

And consequently:

$\Gamma^* = \Gamma_i \cup \Gamma_{i+1}$ and $W^* = A(\Gamma^*)$.

Case 2 $\begin{bmatrix} T_{i+1} \\ X_{i+1} \end{bmatrix} \in A2$ (figure 5.b)

The merging maneuver *i+1* occurs in region $A2$ that is not concerned with the merging maneuver *i*. The merging maneuver *i+1* is not disturbed by the merging maneuver i and

$$W_{i+1} = A(\Gamma_{i+1}) \quad (11.1)$$

The void created by the merging maneuver *i+1* modifies the demand upstream from the merging maneuver *i*. It "stops" the propagation of the shockwave initiated by the merging maneuver *i*, as illustrated in figure 5(b). It follows that W_i is now a transformation of the MBC Γ_i cut off from a certain point and

$$W_i = A(\Gamma_i^{0\to\theta_i}) \quad (11.2)$$

where $\Gamma_i^{0\to\theta_i}$ denotes the remaining part of the MBC.

Thus, $\Gamma^* = \Gamma_i^{0\to\theta_i} \cup \Gamma_{i+1}$ and $W^* = A(\Gamma^*)$.

Case 3 $\begin{bmatrix} T_{i+1} \\ X_{i+1} \end{bmatrix} \in (B \cup C)$ (figure 5.c)

The merging maneuver *i+1* occurs in region B that is concerned with the merging maneuver *i*. This region is congested ($v < v_f$) and it has two consequences. First, it implies that the merging maneuver *i+1* introduces a MBC that "stops" the propagation of the shockwave initiated by the merging maneuver *i*, as illustrated in figure 5.c. Then the solution consists in considering the MBC Γ_i cut off from a point $\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix}$ to its end. Let $\Gamma_i^{0\to\theta_i}$ denote the remaining part of the boundary condition Γ_i. The kinematic wave theory gives $\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix} = P\left(\begin{bmatrix} T_{i+1} \\ X_{i+1} \end{bmatrix}, \Gamma_i \right)$ and

$$W_i = A.\left(\begin{bmatrix} T_i \\ X_i \end{bmatrix} + \Gamma_i^{0\to\theta_i} \right) \quad (12.1)$$

Moreover, according to the kinematic wave theory, the shockwave initiated by the merging maneuver *i+1* is displaced in time and space, as illustrated in figure 6(c) and now starts from the point $A.\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix}$ and:

$$W_{i+1} = A.\left(\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix} + \Gamma_{i+1} \right) \quad (12.2)$$

Thus, $\Gamma^* = \Gamma_i^{0\to\theta_i} \cup \left(\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix} + \Gamma_{i+1} \right)$ and $W^* = A(\Gamma^*)$.

Case 4 $\begin{bmatrix} T_{i+1} \\ X_{i+1} \end{bmatrix} \in O$ (figure 5.d)

The merging maneuver *i+1* occurs in the region *O* that is free of vehicle. The MBC Γ_i intersects Γ_{i+1} at some point $\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix} = \begin{bmatrix} T_{i+1} + \theta_{i+1} \\ X_{i+1} + \chi_{i+1} \end{bmatrix}$ and the resulting MBC corresponds to the minimum of the two initial MBCs being:

- Γ_i ends at this point. If $\Gamma_i^{0\to\theta_i}$ refers to the resulting MBC, then it comes
  $$W_i = A.(\Gamma_i^{0\to\theta_i}) \qquad (13.1)$$
- Γ_{i+1} starts from this point and $\Gamma_{i+1}^{\theta_{i+1}\to end}$ refers to the resulting MBC
  $$W_{i+1} = A.\left(\begin{bmatrix} T_i + \theta_{i+1} \\ X_i + \chi_{i+1} \end{bmatrix} + \Gamma_{i+1}^{\theta_{i+1}\to end} \right) \qquad (13.2)$$

Thus, $\Gamma^* = \Gamma_i^{0\to\theta_i} \cup \left(\begin{bmatrix} T_i + \theta_i \\ X_i + \chi_i \end{bmatrix} + \Gamma_{i+1}^{\theta_{i+1}\to end} \right)$ and $W^* = A(\Gamma^*)$.

These four cases encompass most of the possible interactions between two successive merging maneuvers. The extension of this result to *N* merging maneuvers is non-trivial: for each merging maneuver *i*, one has to take into account every possible interaction with the *N-1* other merging maneuvers. This work performed for the *N* merging maneuvers gives the overall MBCs, denoted by Γ^*. Then the overall shockwave induced by these *N* merging maneuvers can be obtained by $W = A(\Gamma^*)$.
Note that the four cases presented here consider identical merging speed, free-flow speed and acceleration model (and consequently identical MBCs) for every merging maneuver. This assumption can easily be relaxed. The function *A* do not depend on the shape of the MBC. Thus the results presented here still holds with a combination of different MBC's shapes. In other term, the results presented here are independent of the acceleration model and can be extend to any bounded acceleration model.

This section has detailed the analytical solution of the kinematic wave model with vehicles that insert with low speed. It leads to the explicit expression of the number of vehicles impacted by one or more merging maneuvers. It also gives the analytical formulation of the shockwave(s) initiated by one (or multiple) merging maneuver(s).
In Laval (Laval, 2005), such merging maneuvers are suspected to be the cause of flow drop. In the next section, we study some application cases to explore this assumption. We show that low speed merging maneuvers create voids that cannot be cleared and that reduce the capacity.

3. Application cases

3.1 General definition of the flow

Let's consider the function *N(t,x)* representing the vehicle number (numbered from a reference vehicle) as a function of time and space. *N* is a discrete function that increases with time and decreases with space. Note that in the following, *n* is not necessarily an integer and can correspond to a quantum of vehicle. The flow is defined as a difference in number of vehicles passing through a location X over a period of time ΔT . If the period starts at the time T_i, we have

$$Q(X,Ti,\Delta T)=\frac{N(X,Ti+\Delta T)-N(X,Ti)}{\Delta T}=\frac{\Delta N(X,Ti,\Delta T)}{\Delta T}$$

Here we study a scenario with a single merging maneuver. X =0 at the merging location. The aim of this section is to measure the variation in flow between a fictitious reference vehicle (vehicle *0*) traveling at the free-flow speed v_f and its n^th (vehicle *n*). Let $\Delta T_0^N(X)$ denote the bumper to bumper time headway between vehicle *0* and vehicle *n* at the location X. This is illustrated in figure 6(a) and the flow equals:

$$Q_{0\to N}(X)=\frac{N}{\Delta T_0^N(X)} \tag{14}$$

Previously, it was demonstrated that a single merging maneuver involves a variation of flow in space and time. We now quantify this variation to estimate a flow-drop around a single merging maneuver.

3.2 Calculation of the flow-drop for a single merging maneuver

Consider here the point $\begin{bmatrix} T \\ X \end{bmatrix}$: *N(T,X)* represents the number of vehicle passing through X at the time T. several cases can be identified :

- If $\begin{bmatrix} T \\ X \end{bmatrix}\in A$; it is straightforward that $Q_{0\to N}(X)=\dfrac{N(T,X)}{\Delta T_0^N(X)}=\dfrac{Q_A.\Delta T_0^N(X)}{\Delta T_0^N(X)}=Q_A$

- If $\begin{bmatrix} T \\ X \end{bmatrix}\in O$; N(T,X)=0 and $Q_{0\to N}(X)=0$

- If $\begin{bmatrix} T \\ X \end{bmatrix}\in (B\cup C)$; the calculation is not straightforward because the flow varies now in time and space. Then it is convenient to introduce two specific points, as illustrated in figure 6(b). The first point M1 corresponds to the projection of the point W_N on the MBC Γ_0: $M1=\begin{bmatrix} \theta_1 \\ \chi_1 \end{bmatrix}=P(W_N,\Gamma_0)$. The second point M2 corresponds to the projection of the point $\begin{bmatrix} T \\ X \end{bmatrix}$ on the MBC Γ_0:

$$M2 = \begin{bmatrix} \theta_2 \\ \chi_2 \end{bmatrix} = P\left(\begin{bmatrix} T \\ X \end{bmatrix}, \Gamma_0\right)$$ (see in figure 6.b).

This scenario excludes any entrance or exit lane-changing maneuvers between the reference vehicle and its N(T,X)th follower. As a consequence, the conservation law holds between a location upstream ($X = X_u$) and downstream ($X = X_d$) the merging location X_m:

$$N(T,X) = Q_A.\Delta T_0^N(X_u) = Q_A\left(\Delta T_0^N - \left((\theta_2 - \theta_1) - \frac{\chi_2 - \chi_1}{v_f}\right)\right) \qquad \textit{[upstream, } X<X_m\textit{]}$$

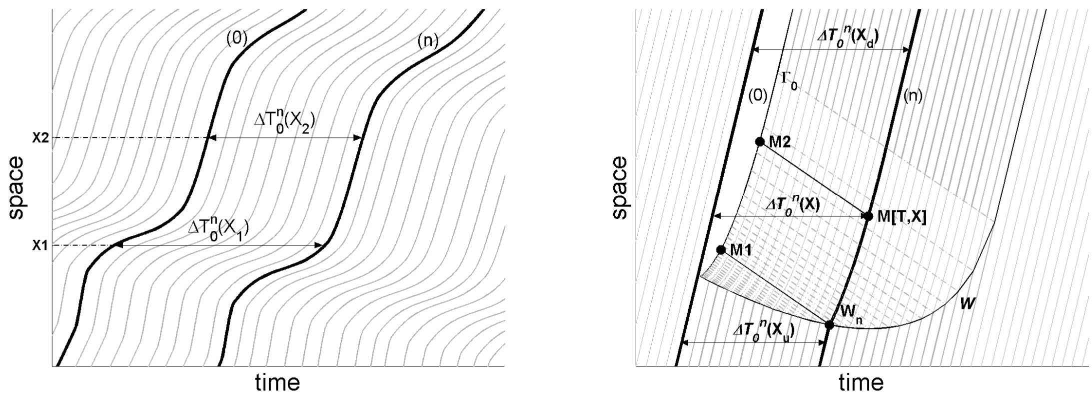
$$N(T,X) = Q_C.\Delta T_0^N(X_d) = Q_C\left(\Delta T_0^N - \left(\theta_2 - \frac{\chi_2}{v_f}\right)\right) + 1 \qquad \textit{[downstream, } X>X_m\textit{]}$$

The flow conservation law combined with the previous definition of the flow gives

$$Q_{0\to N}(X) = \frac{N(T,X)}{\Delta T_0^N(X)} = Q_A.(1-\alpha) = Q_C.(1-\beta) + \frac{1}{Q_A.\Delta T_0^N(X)} \qquad (15)$$

where α and β depends on X.

- $\alpha(X) = \frac{1}{\Delta T_0^N(X)}.\left((\theta_2 - \theta_1) - \frac{\chi_2 - \chi_1}{v_f}\right)$ represents the relative delay of the vehicle N at the location X.
- $\beta(X) = \frac{1}{\Delta T_0^N(X)}.\left(\theta_2 - \frac{\chi_2}{v_f}\right)$ represents the relative delay of the reference vehicle at the location χ_2.



(a) **(b)**

Figure 6 (a) definition of the flow based on time intervals
(b) variation of the flow around a single merging maneuver

Merging maneuvers at low speed introduce voids within the flow. This heterogeneity causes a new distribution of the flow in space and time, that is represented by the coefficients α and β. Now consider the flow Q_A upstream the merging point X_m and compare this reference flow to the flow measured over the zone $B \cup C$. Then the flow drop between the reference vehicle and its N^{th} follower at the location X is measured as:

$$\Delta Q_{0\to N}(X) = \frac{Q_A - Q_{0\to N}(X)}{Q_A} = \alpha = 1 - \frac{Q_C}{Q_A}(1-\beta) - \frac{1}{Q_A.\Delta T_0^N(X)}$$

According to this definition, coefficient α characterizes the flow drop in the region B and C. From this definition, they are several cases with simple results:

- along Γ_0: $\begin{bmatrix}\theta_1\\ \chi_1\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ consequently $\alpha = 1$ and $\Delta Q_{0\to N}(X) = 1$. The flow drop is maximum: this result is straightforward since there is no vehicle between the fictitious reference trajectory and Γ_0.
- along W: $\begin{bmatrix}\theta_1\\ \chi_1\end{bmatrix} = \begin{bmatrix}\theta_2\\ \chi_2\end{bmatrix}$ so $\alpha = 0$. In this case, the relative delay of any vehicle is zero when it crosses W. This result is intuitive since the flow drop cannot be measured over a period that encompasses the global disturbance.
- In the zone C: $\begin{bmatrix}\theta_2\\ \chi_2\end{bmatrix} = \begin{bmatrix}\Theta\\ \mathrm{X}\end{bmatrix}$. It follows $\Delta Q_{0\to N}(X) = \frac{1}{\Delta T_0^N}.\frac{Q_C}{Q_A}.\left(\Theta - \frac{\mathrm{X}}{v_f}\right)$. Note that the flow drop mainly depends on the length of the period $\Delta T_0^N(X)$.

The figure 7 summarizes the variations of the flow-drop versus the space and the time.

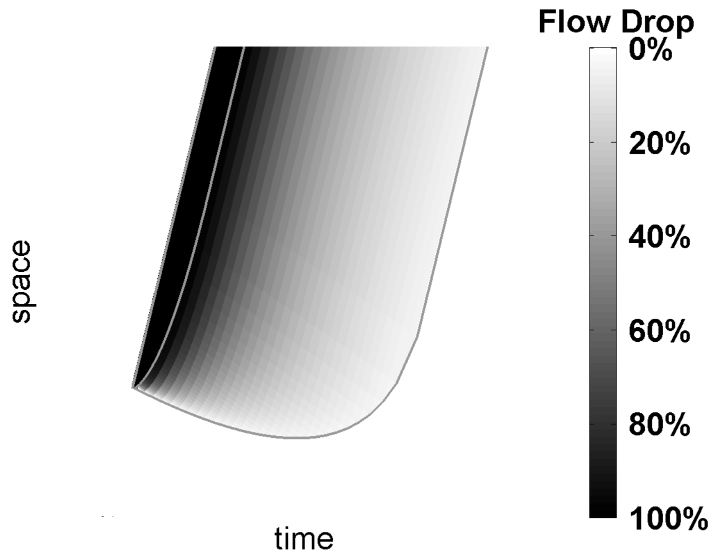


Figure 7 flow-drop around a single merging maneuver

3.3 Calculation of the capacity-drop for *N* merging maneuvers at a single location

The previous paragraph shows that low speed merging maneuvers create heterogeneities, but it should be noted that they also create voids within the traffic stream. They represent some "empty areas" that reduce the maximum flow. This phenomenon, called the capacity drop, can be represented by the ratio $\gamma = \frac{Q_C'}{Q_C}$, where Q_C is the original downstream capacity and Q_C' is the new downstream capacity.

To estimate this ratio, consider here a large number of merging maneuvers, performed at a constant rate $Q_i = \frac{1}{T_i}$ and at a single location *X*=0. The new capacity is the flow passing through a point $X=X_d$ far downstream of the merging maneuver. To estimate the capacity drop, we focus on the time period between two successive merging maneuvers at the location $X=X_d$ and we measure the length of the sub-period that is free of vehicle.

Then two cases must be distinguished, depending on whether the merging maneuvers interact or not.

First case: the merging maneuvers do not interact (figure 8(a))
The period between two successive merging maneuvers *n* and *n+1* can be divided into two sub-periods: the first sub-period, free of vehicle (void), lasts $\left(\Theta - \mathrm{X}/v_f\right)$. The second sub-period, that can receive a maximum flow equals to the capacity, lasts $T_i - \left(\Theta - \mathrm{X}/v_f\right)$. The two sub-periods considered finally lead to a capacity drop:

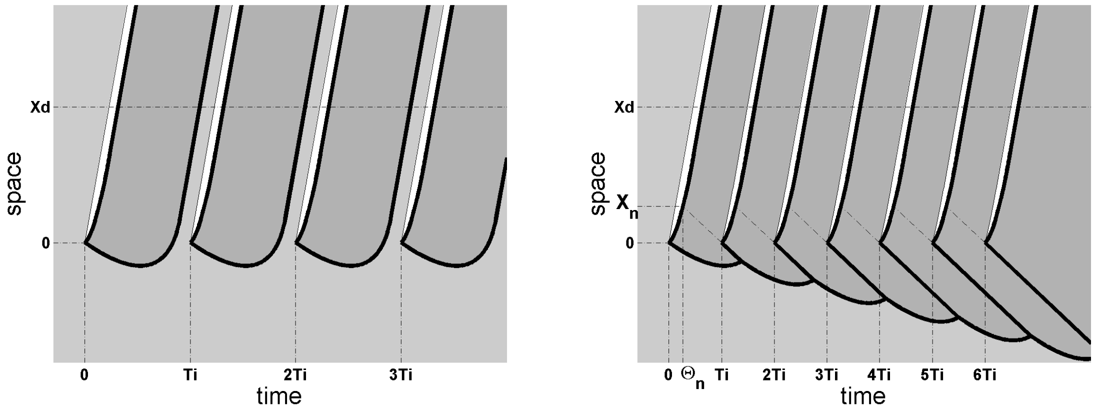
$$\gamma = 1 - \frac{\left(\Theta - \mathrm{X}/v_f\right)}{T_i} + \frac{Q_i}{Q_c} \qquad (16)$$

It should be noted that if the merging maneuvers do not interact, then the congestion does not propagate far upstream the merge: this is a local congestion. The analytical expressions of the most upstream points reached by the shockwaves depend on the upstream demand, the FD parameters and the merging maneuver speed.

Second case: the merging maneuvers interact (figure 8(b))
The period between two successive merging maneuvers can be divided into two sub-periods as well. Because of the interaction, the sub-period free of vehicle is shortened and equals $\left(\Theta_n - \mathrm{X}_n/v_f\right)$, where $\begin{bmatrix}\Theta_n \\ \mathrm{X}_n\end{bmatrix} = P(\begin{bmatrix}T_{n+1} \\ X_{n+1}\end{bmatrix}, \Gamma_n)$ and X_n depend on the merging maneuver rate. As a consequence, the capacity drop is:

$$\gamma = \frac{T_i - \left(\Theta_n - \mathrm{X}_n/v_f\right)}{T_i} + \frac{Q_i}{Q_c} \qquad (17)$$

It should be noted that if the merging maneuvers interact, then the congestion does propagate far upstream the merge: this is a global congestion. The analytical expression of the shockwave propagation's speed depends on the upstream demand, the FD parameters, the merging maneuver speed and the merging maneuver's rate.



(a) **(b)**

Figure 8 (a) capacity-drop without interaction (b) capacity-drop with interactions

4. Conclusion

This paper examined the analytical solution of low speed merging maneuvers within a free-flow stream. It relies on a simple first order traffic model where some moving boundary conditions are introduced. A single MBC can trigger a local congestion, under the condition that combines the upstream demand and the FD parameters. If this condition is satisfied, a shockwave propagates in time and space, and the analytical solution of the model gives that its trajectory is a simple linear transformation of the MBC. The transformation depends on the FD parameter and the demand upstream of the merging maneuver. Consequently, if the acceleration model is known, the impact of a single low speed merging maneuver is known as well. These results have then been extended to multiple merging maneuvers. Merging maneuvers can interact and it modifies the MBCs. The new shockwaves are still a linear transformation of these new boundary conditions. Finally, we show that merging maneuvers create voids that cannot be cleared. These empty areas decrease the capacity downstream the merging maneuver and it can be related to a capacity drop. The typical scenario of multiple merging maneuvers at a single location is analyzed and a general analytical formulation of the capacity drop is proposed.

This work still requires further research to deepen the knowledge about the impact of low speed merging maneuver. First, a sensitivity analysis could help to understand the important factors that trigger and propagate the congestion. Also, this study has assumed some simplifying assumptions (the upstream demand is constant; every vehicle has the same FD; every vehicles inserts at the same speed; the merging maneuver rate is constant; etc.) and the results may be different in more general conditions. And more importantly, some indicators must be developed to estimate the macroscopic impact of such maneuvers. They can be based on travel time, delays of total time spent on the network, or alternative measures. On the basis of this work, more complex indicators can be built like the total number of vehicle impacted; the time to clear the (local or global) congestion, the most upstream point reached by the shockwave(s). Furthermore, results from this modeling framework need to be compared to on-field data.

The results presented in this paper can also be applied for operational applications. These analytical results can be the basis of a tool for traffic road manager. Applied for rerouting strategies, the results of presented here can help to control the demand upstream of a merge and avoid global congestion. By applying on ramp-metering strategies, these results can help to control a dynamic the merging maneuver rate online, limiting the merging maneuver's impact on the major stream. Also, it can provide the minimal merging maneuver speed for minimizing the impact of low speed merging maneuvers.

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